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Foundations · market mechanics from the ground up

The option price: from a coin toss to the Black-Scholes formula

Chapter 6 of 10

The price of an option is not an expected value and not a forecast. It is the price of a portfolio of index and loan that replicates the option's payoff exactly. This chapter derives that completely on a single coin toss, builds the binomial tree from it, shows the transition to the Black-Scholes formula and explains implied volatility, skew and the premium that option sellers earn on average.

Field	Value
Series	Futures, options and who makes the price
Scope	Chapter 6 of 10
Type	Explanation with worked examples, not a measurement
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In one sentence: An option costs as much as the portfolio of index and loan that pays out exactly the same in the end, and because that portfolio is built without any assumption about the direction of the market, the option price contains no view on it, only an assumption about fluctuation.

What this chapter is: a derivation with worked examples. The transition from the tree to the formula is shown, not strictly proven. The numbers on the skew are example values. The numbers on the volatility premium come from our own measurement and are linked. **Not a trading recommendation.**

Chapter 5 left open why the call with strike 7,000 costs exactly 128.09 points. Here that price is derived.

1. The obvious calculation, and why it is wrong

The index is at 7,000. To simplify, there are only two possibilities in one month: it is at 7,350 (plus 5 %) or at 6,650 (minus 5 %). The interest rate is zero. A call with strike 7,000 then pays 350 or nothing.

What is it worth today? The obvious answer is the expected value: payoff times probability.

- Whoever regards both cases as equally likely computes $0.5 \times 350 = 175$.
- Whoever reckons with a rise at 70 % arrives at $0.7 \times 350 = 245$.

On that basis everyone would have his own price, depending on his view. In fact there is only one correct price, and to find it one does not need to know the probability at all.

2. The replication: a portfolio that pays the same

The idea: from two things whose price we know we build a portfolio that is worth exactly as much as the call in both cases. The two things are the index itself and a loan.

The portfolio consists of Δ units of the index, financed partly with a loan of size B. It should be worth as much as the call in both cases:

$$\begin{aligned} \text{Index rises:} \quad & \Delta \times 7{,}350 - B = 350 \\ \text{Index falls:} \quad & \Delta \times 6{,}650 - B = 0 \end{aligned}$$

Step 1: subtract the second equation from the first. The loan drops out:

$$\Delta \times (7{,}350 - 6{,}650) = 350 \quad \Longrightarrow \quad \Delta = \frac{350}{700} = 0.5$$

Step 2: determine the loan from the second equation.

$$B = 0.5 \times 6{,}650 = 3{,}325$$

Step 3: what does this portfolio cost today? Half an index unit costs 3,500, of which 3,325 is borrowed:

$$0.5 \times 7{,}000 - 3{,}325 = 175$$

The check. If the index rises, the half unit is worth 3,675, less 3,325 of loan leaves 350. If it falls, it is worth 3,325, less the loan leaves zero. The portfolio pays exactly like the call in both cases.

Two things that pay the same in every case must cost the same today. The call is worth 175.

3. Why it has to be this price

Suppose the call trades at 200 because many believe in rising prices. A dealer sells the call and rebuilds it for 175:

Trade today	Cash flow today
Sell call	+200
Buy 0.5 index units	−3,500
Take out loan	+3,325
Sum	+25

In the end his portfolio offsets the obligation from the call exactly in both cases. The 25 points remain with him, without risk. This works until the price has fallen to 175. If the call is below 175, it runs the other way.

The probability of a rise did not appear in this calculation. Whether it is 50 % or 70 % does not change the price. The reason: the market's view on direction is already in the index level of 7,000. Whoever is more optimistic than the market should buy the index. The option adds nothing to that.

What does determine the price is the **distance** between the two possibilities. If they were at plus and minus 10 %, the call would be twice as expensive. The option price depends on fluctuation, not on direction.

4. The general formula

The same calculation with letters. The index is at S and jumps to S_u or S_d. The call is then worth C_u or C_d.

$$\Delta \cdot S_u - B = C_u$$
$$\Delta \cdot S_d - B = C_d$$

Subtracting gives the number of units:

$$\Delta = \frac{C_u - C_d}{S_u - S_d}$$

That is the **delta**: the difference of the option values divided by the difference of the index levels. It says how much index is needed to rebuild the option. In chapter 8 it becomes the dealer's hedge.

The loan follows from the second equation: $B = \Delta \cdot S_d - C_d$. The price of the portfolio today is $\Delta \cdot S - B$. Substituted:

$$C = \Delta \cdot (S - S_d) + C_d = \frac{S - S_d}{S_u - S_d} \cdot (C_u - C_d) + C_d$$

Calling the fraction q gives a simple form:

$$C = q \cdot C_u + (1 - q) \cdot C_d, \qquad q = \frac{S - S_d}{S_u - S_d}$$

In the example $q = (7{,}000 - 6{,}650)/(7{,}350 - 6{,}650) = 0.5$ and $C = 0.5 \times 350 = 175$.

The formula looks like an expected value with the probability q. But q is not a forecast. It is a computed quantity that follows from the index levels alone: the probability at which the index itself earns nothing on average. It is called the **risk-neutral probability**. With an interest rate r little changes: the numerator of q then

contains $S \cdot e^{rT}$ instead of S, and the result is discounted with e^{-rT}.

5. Two steps: the replication has to be adjusted

A single jump is too coarse. With two steps the picture already becomes richer. The index rises or falls per step by the factor 1.035 or 1/1.035:

	Start	after step 1	after step 2	Call payoff
risen twice			7,498.57	498.57
risen once		7,245.00		
middle	7,000		7,000.00	0
fallen once		6,763.29		
fallen twice			6,534.57	0

The calculation runs from back to front, each node with the formula from section 4. Here $q = 0.4914$.

- **After a rise (7,245):** the call is worth $0.4914 \times 498.57 = 245.00$.
- **After a fall (6,763.29):** both following states pay nothing. The call is worth 0.
- **At the start:** $0.4914 \times 245.00 = 120.39$.

More important than the price is what happens to the delta:

Node	Delta
Start (7,000)	0.51
after a rise (7,245)	1.00
after a fall (6,763.29)	0.00

Whoever rebuilds the call holds 0.51 index units at the start. If the index rises, he has to increase to a whole unit. If the index falls, he has to sell everything. **The replication is not a one-off trade. It has to be adjusted after every move**, and always in the direction of the move: buy after a rise, sell after a fall.

That is the core of everything that follows in part III. A dealer who has sold a call rebuilds it in exactly this way to offset his risk. His trading in the index is thereby dictated by the price path.

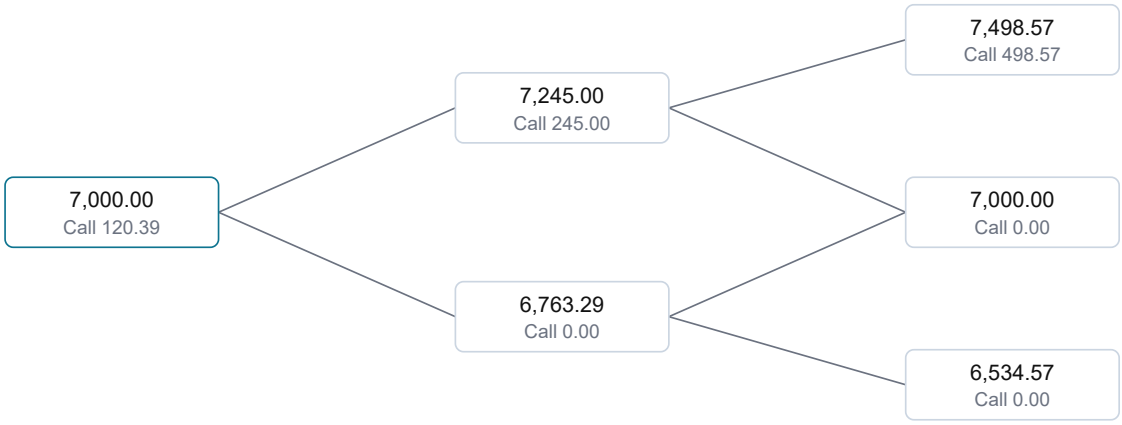


Fig. 1: The tree with two steps. At each node the index level is on top and below it the value of the call with strike 7,000.

6. Many steps

The more steps the tree has, the more finely it maps the possible paths of the index. The size of the steps is tied to the fluctuation one reckons with. For a volatility σ per year and steps of length Δt one chooses

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = \frac{1}{u}$$

Smaller steps in time thus get smaller jumps in price, in the ratio of the square root. That is the same root as in chapter 3: fluctuation grows with the square root of time.

For the call from chapter 5 (index 7,000, strike 7,000, 30 days, 16 % volatility, zero interest) the tree gives:

Number of steps	Price of the call
1	160.52
2	113.51
5	134.61
10	124.93
50	127.45
100	127.77
1,000	128.05

The price settles down. If the number of steps goes to infinity, it lands at 128.09. That is the price from chapter 5.

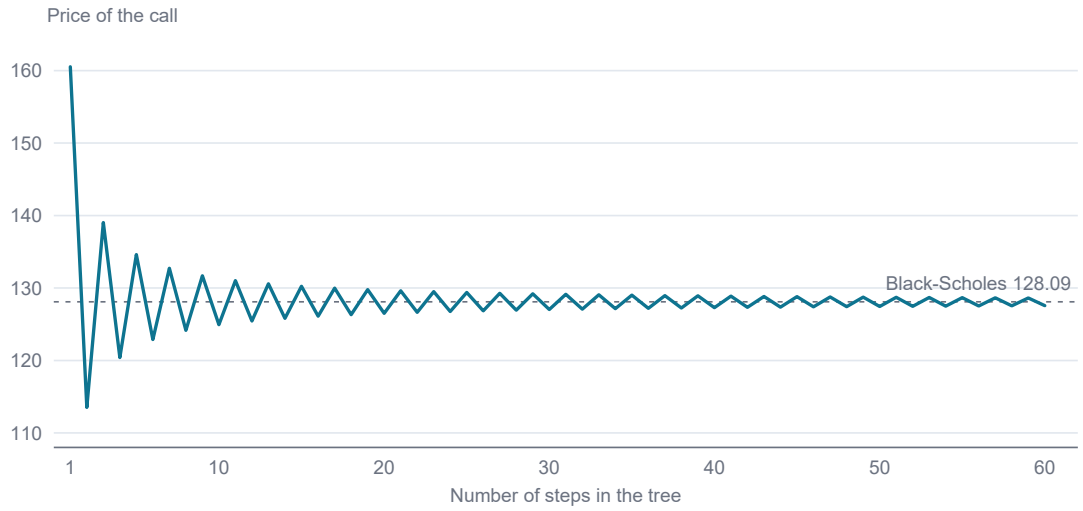


Fig. 2: Price of the call in the tree for 1 to 60 steps. The horizontal line is the limit 128.09 that the Black-Scholes formula gives.

7. The Black-Scholes formula

For the limiting case of infinitely many steps there is a closed formula. Fischer Black, Myron Scholes and Robert Merton published it in 1973. For a call:

$$C = S \cdot N(d_1) - K \cdot e^{-rT} \cdot N(d_2)$$

with

$$d_1 = \frac{\ln(S/K) + \left(r + \tfrac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

Every quantity has a meaning:

- $N(\cdot)$ is the distribution function of the normal distribution. It gives a probability between 0 and 1 for every number.
- $\sigma\sqrt{T}$ is the fluctuation over the whole term. At 16 % per year and 30 days that is $0.16 \times \sqrt{30/365} = 4.59\ \%$.
- $N(d_2)$ is the risk-neutral probability that the option ends in the money. So $K \cdot e^{-rT} \cdot N(d_2)$ is today's value of the strike that is paid on exercise.
- $S \cdot N(d_1)$ is today's value of what is received on exercise.
- $N(d_1)$ is at the same time the **delta**: the number of index units in the replication, as in the tree.

So the formula is the same as in the tree: number of units times index minus loan. Only instead of two states there are infinitely many behind it.

Worked example. Index 7,000, strike 7,000, 30 days, 16 % volatility, zero interest:

$$\begin{aligned} \sigma\sqrt{T} &= 0.16 \times \sqrt{30/365} = 0.04587 \\ d_1 &= \frac{\ln(1) + \tfrac{1}{2} \times 0.16^2 \times 30/365}{0.04587} = 0.02294 \\ d_2 &= 0.02294 - 0.04587 = -0.02294 \\ N(d_1) &= 0.50915, \qquad N(d_2) = 0.49085 \\ C &= 7{,}000 \times 0.50915 - 7{,}000 \times 0.49085 = 128.09 \end{aligned}$$

The price of the put follows from put-call parity in chapter 5.

What the formula assumes: a volatility that does not change, a price without jumps, continuous adjustment of the replication at no cost. None of these assumptions holds exactly. Sections 9 and 10 show how the market deals with that.

8. Implied volatility

Of the five quantities in the formula four are known: index, strike, term and interest rate. The fifth, the volatility until expiry, nobody knows.

Dealers therefore turn the formula round. They do not ask: "What price follows from the volatility?" They ask: "What volatility do I have to put in so that the price comes out at which the option is trading right now?" That number is called **implied volatility**.

Worked example. The call does not trade at 128.09 but at 140. Substituting and trying until it fits gives an implied volatility of 17.49 %. As a rule of thumb: one volatility point more makes this option about 8 points dearer. So 12 points more in price are about 1.5 volatility points.

Implied volatility is thus another way of writing the price. It makes options comparable: 140 points for a call say little, 17.5 % volatility can be compared with yesterday, with other strikes and with other markets. The best-known value of this kind is the VIX, the implied volatility of the S&P 500 for the next 30 days.

9. The skew: insurance costs extra

If the formula were exactly right, all options on the same index with the same expiry would have to have the same implied volatility. That is not the case. For stock indices the following has held since the crash of 1987:

- Puts with a low strike, which protect against sharp losses, have a **higher** implied volatility.
- Calls with a high strike have a **lower** one.

This slope is called the **skew**. What it means in points is shown by an example with assumed volatility values:

Option	Price at 16 % volatility	Assumed market volatility	Market price
Put, strike 6,650 (5 % below the index)	20.71	20 %	39.67
Call, strike 7,350 (5 % above the index)	24.24	13 %	11.94

Protection against a fall of more than 5 % here costs almost twice what the formula gives with a uniform volatility. Two reasons work together:

- **Demand.** Funds and pension funds permanently buy puts as protection. Whoever sells them carries a risk that is hard to pass on, and demands more for it.
- **The assumptions fit worse to the downside.** Markets fall faster than they rise, sometimes in jumps, and fluctuation increases in sell-offs. The replication from section 5 then works worse than the formula assumes.

10. The volatility premium: implied lies above realised

Implied volatility is a price, and a price for insurance contains a mark-up. That is why implied volatility lies on average above the fluctuation that then actually occurs.

We have measured that. For 2015 to 2022 the fluctuation of the preceding 20 days was in the median only 0.76 (S&P 500), 0.84 (Nasdaq 100) and 0.74 (Dow) of the implied volatility. One consequence: the daily range priced in by the option market held the closing price on 80.1 % of days. With a normal distribution and no mark-up 68.3 % would be expected (to the study).

This difference is called the **volatility risk premium**. It is the pay of those who sell options, and it is not free money. The seller regularly takes in small amounts in quiet times and loses a great deal at once in a crash. How this shows up day by day in a dealer's book is worked through in chapter 8.

Common misconception

> **Common misconception:** "Implied volatility is the market's forecast of the coming fluctuation." It is a price, converted into a volatility. That price contains the expected fluctuation and in addition a mark-up for the risk the seller carries. That is why it lies on average above the fluctuation that then occurs. Whoever concludes from a high VIX that the market "expects" exactly this move systematically overestimates it.

What you should understand now

- The price of an option is the price of the portfolio of index and loan that replicates its payoff. The probability of a rise does not appear in it.
- The number of index units in the replication is the delta: $\Delta = (C_u - C_d)/(S_u - S_d)$. It changes with the price, so the replication has to be adjusted continuously.
- With more and more steps the tree turns into the Black-Scholes formula. Its only unknown quantity is volatility.
- Implied volatility is the market price converted into a volatility. It differs by strike. Downside protection is dearer (skew).
- Implied volatility lies on average above realised volatility. Option sellers earn this premium for carrying crash risk.

Limits of this account

- **No proof.** That the tree converges to the formula is shown with numbers, not proven mathematically. The proof needs tools that are not assumed here.
- **Model assumptions.** Constant volatility, no jumps, free and continuous trading, no interest in the worked example. The skew is the market's visible correction for these assumptions.
- **Example values.** The volatility values in section 9 (20 % and 13 %) are chosen for illustration, not measured. How steep the skew is changes all the time.
- **Own measurement.** The ratio of realised to implied volatility comes from 2015 to 2022 and compares with the fluctuation of the 20 days before.
- **Not covered:** options with early exercise, dividends in the tree and models with changing volatility.

Continue with chapter 7: the Greeks, that is, how strongly the option price reacts to price, time and volatility.

An explanation of market mechanics with worked examples. Not investment advice and not a trading recommendation. Trading futures and options involves the risk of loss up to and beyond the capital employed.